m*-FUZZY BASICALLY DISCONNECTED SPACES IN SMOOTH FUZZY TOPOLOGICAL SPACES

B. Amudhambigai
rbamudha@yahoo.co.in

M.K. Uma and E. Roja
Department of Mathematics
Sri Sarada College for Women, Salem-16
Tamil Nadu, India

Abstract
In this paper, the concepts of m* r-fuzzy $\tilde{g}$-open $F_\sigma$ sets and m*-fuzzy basically disconnected spaces are introduced in the sense of Sostak [9] and Ramadan [6]. Some interesting properties and characterizations are studied. Tietze extension theorem for m*-fuzzy basically disconnected spaces has been discussed as in [12].

Key words
m* r-fuzzy $\tilde{g}$-open $F_\sigma$ sets, m*-fuzzy basically disconnected spaces, Lower m*-fuzzy continuous functions and upper m*-fuzzy continuous functions.

2000 Mathematics Subject Classification : 54A40, 03E72.

1. Introduction and Preliminaries
The concept of fuzzy set was introduced by Zadeh [13] in his classical paper. Fuzzy sets have applications in many fields such as information [8] and control [10]. In 1985, Sostak [9] introduced a new form of topological structure. In 1992, Ramadan [6] studied the concept of smooth fuzzy topological spaces. The concept of $\tilde{g}$-open set was discussed by Rajesh and ErdalEkici [5]. The concept of fuzzy basically disconnected spaces was introduced and studied in [11]. The notions of m-structures, m-spaces, and m-continuity were introduced by Popa and Noiri [3, 4]. The concepts of r-fuzzy $G_\delta$ set and r-
fuzzy $F_\sigma$ set were introduced in [2]. In this paper, the concepts of $m^*$-r-fuzzy $\tilde{g}$-open $F_\sigma$ sets and $m^*$-fuzzy basically disconnected spaces are introduced in the sense of Sostak [9] and Ramadan [6]. Some interesting properties and characterizations are studied. Tietze extension theorem for $m^*$-fuzzy basically disconnected spaces has been discussed as in [11].

Throughout this paper, let $X$ be a nonempty set, $I = [0,1]$ and $I_0 = (0,1]$. For $\xi \in I$, $T(\xi) = \xi$ for all $x \in X$.

Definition 1.1 [9] A function $T : I^X \rightarrow I$ is called a smooth fuzzy topology on $X$ if it satisfies the following conditions:

1. $T(0) = T(1) = 1$.
2. $T(\mu_1 \wedge \mu_2) \geq T(\mu_1) \wedge T(\mu_2)$ for any $\mu_1, \mu_2 \in I^X$.
3. $T(\bigvee_{j \in \Gamma} \mu_j) \geq \bigwedge_{j \in \Gamma} T(\mu_j)$ for any $\{ \mu_j \}_{j \in \Gamma} \in I^X$.

The pair $(X, T)$ is called a smooth fuzzy topological space.

Remark 1.1 Let $(X, T)$ be a smooth fuzzy topological space. Then, for each $r \in I_0$, $T_r = \{ \mu \in I^X : T(\mu) \geq r \}$ is Chang’s fuzzy topology on $X$.

Definition 1.2 [7] Let $(X, T)$ be a smooth fuzzy topological space. For each $\lambda \in I^X$, $r \in I_0$, an operator $C_T : I^X \times I_0 \rightarrow I^X$ is defined as follows: $C_T(\lambda, r) = \bigwedge \{ \mu : \mu \geq \lambda, T(\mu) \geq r \}$. For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

1. $C_T(0, r) = 0$.
2. $\lambda \leq C_T(\lambda, r)$.
3. $C_T(\lambda, r) \vee C_T(\mu, r) = C_T(\lambda \vee \mu, r)$.
4. $C_T(\lambda, r) \leq C_T(\lambda, s)$, if $r \leq s$.
5. $C_T(C_T(\lambda, r), r) = C_T(\lambda, r)$.

Proposition 1.1 [7] Let $(X, T)$ be a smooth fuzzy topological space. For each $\lambda \in I^X$, $r \in I_0$, an operator $I_T : I^X \times I_0 \rightarrow I^X$ is defined as follows: $I_T(\lambda, r) = \bigvee \{ \mu : \mu \leq \lambda, T(\mu) \geq r \}$. For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

1. $I_T(1 - \lambda, r) = 1 - C_T(\lambda, r)$. 

92
\( m^*\)-FUZZY BASICALLY DISCONNECTED

(2) \( I_T(\hat{1}, r) = \hat{1} \).

(3) \( \lambda \geq I_T(\lambda, r) \).

(4) \( I_T(\lambda, r) \land I_T(\mu, r) = I_T(\lambda \land \mu, r) \).

(5) \( I_T(\lambda, r) \geq I_T(\lambda, s), \) if \( r \leq s \).

(6) \( I_T(I_T(\lambda, r), r) = I_T(\lambda, r) \).

Definition 1.3 [2] Let \((X, T)\) be a smooth fuzzy topological space, \( r \in I_0 \). For any \( \lambda \in I^X \) and \( r \in I_0 \), \( \lambda \) is called

(a) an \( r \)-fuzzy \( G_\delta \) set if \( \lambda = \bigwedge_{i=1}^{\infty} \lambda_i \) where each \( \lambda_i \) is such that \( T(\lambda_i) \geq r \).

(b) an \( r \)-fuzzy \( F_\sigma \) set if \( \lambda = \bigvee_{i=1}^{\infty} \lambda_i \) where each \( 1 - \lambda_i \) is such that \( T(1 - \lambda_i) \geq r \).

Definition 1.4 [1] Let \((X, T)\) be a smooth fuzzy topological space. For \( \lambda \in I^X \) and \( r \in I_0 \), \( \lambda \) is called

(1) \( r \)-fuzzy \( \hat{g} \)-closed if \( C_T(\lambda, r) \leq \mu \) whenever \( \lambda \leq \mu \) and \( \mu \) is \( r \)-fuzzy semiopen. The complement of an \( r \)-fuzzy \( \hat{g} \)-closed set is said to be an \( r \)-fuzzy \( \hat{g} \)-open set.

(2) \( r \)-fuzzy \( *g \)-closed if \( C_T(\lambda, r) \leq \mu \) whenever \( \lambda \leq \mu \) and \( \mu \) is \( r \)-fuzzy \( \hat{g} \)-open. The complement of an \( r \)-fuzzy \( *g \)-closed set is said to be an \( r \)-fuzzy \( *g \)-open set.

(3) \( r \)-fuzzy \( g \)-semiclosed (briefly \( r-fgs \)-closed) if \( SC_T(\lambda, r) \leq \mu \) whenever \( \lambda \leq \mu \) and \( \mu \) is \( r \)-fuzzy \( *g \)-open. The complement of an \( r \)-fuzzy \( g \)-semiclosed set is said to be an \( r \)-fuzzy \( g \)-semiopen set (briefly \( r-fgs \)-open set).

(4) \( r \)-fuzzy \( \bar{g} \)-closed if \( C_T(\lambda, r) \leq \mu \) whenever \( \lambda \leq \mu \) and \( \mu \) is \( r \)-fuzzy \( \bar{g} \)-open. The complement of an \( r \)-fuzzy \( \bar{g} \)-closed set is said to be an \( r \)-fuzzy \( \bar{g} \)-open set.

Definition 1.5 [3, 4] A subfamily \( m_X \) of the power set \( P(X) \) of a nonempty set \( X \) is called a minimal structure (briefly, \( m \)-structure) on \( X \) if \( \phi \in m_X \) and \( X \in m_X \). By \((X, m_X)\), we denote a nonempty subset \( X \) with a minimal structure \( m_X \) on \( X \) and call it an \( m \)-space. Each member of \( m_X \) is said to be \( m_X \)-open (or briefly, \( m \)-open) and the complement of an \( m_X \)-open set is said to be \( m_X \)-closed (or briefly, \( m \)-closed).

2. \( m^* \)-FUZZY BASICALLY DISCONNECTED SPACES
In this section, the concept of m*-r-fuzzy $g\circ$-open $F_\sigma$ sets and m*-fuzzy basically disconnected spaces are introduced. Some interesting properties and characterizations are studied.

Definition 2.1 Let $X$ be a nonempty set and $I^X$ be a collection of all fuzzy sets in $X$. A subfamily $m_X$ of $I^X$ is called a minimal structure (briefly, m-structure) on $X$ if $\emptyset \in m_X$ and $I \in m_X$.

Notation 2.1 Let $(X, T)$ be a smooth fuzzy topological space, $r \in I_0$.

(1) The family of all r-fuzzy $g\circ$ open sets in $(X, T)$ is denoted by $g\circ O(X, T)$.

(2) The family of all r-fuzzy $F_\sigma$ sets in $(X, T)$ is denoted by $F_\sigma(X, T)$.

Definition 2.2 Let $(X, T)$ be a smooth fuzzy topological space, $r \in I_0$. Then the collection of the families $g\circ O(X, T)$ and $F_\sigma(X, T)$ which is finer than the smooth fuzzy topology $T$ on $X$ is a minimal* structure (briefly, m*-structure) on $X$, denoted by $m_X^*$. A nonempty set $X$ with an m*-structure $m_X^*$ on $X$, denoted by $(X, m_X^*)$ (or briefly, $(X, m^*)$) and it is called as an m*-smooth fuzzy space. Each member of $m_X^*$ is said to be m*-r-fuzzy $g\circ$-open $F_\sigma$ and the complement of m*-r-fuzzy $g\circ$-open $F_\sigma$ set is said to be m*-r-fuzzy $g\circ$-closed $G_\delta$.

Definition 2.3 A minimal structure $m_X^*$ on a nonempty set $X$ is said to have property B if the union of any family of m*-r-fuzzy $g\circ$-open $F_\sigma$ sets belonging to $m_X^*$ belongs to $m_X^*$, $r \in I_0$.

Definition 2.4 Let $(X, T)$ be a smooth fuzzy topological space with an m*-structure $m_X^*$ determined by $T$ and let $m_X^*$ satisfy property B. For any $\lambda \in I^X$ and $r \in I_0$, the m*-r-fuzzy $g\circ G_\delta$-closure of $\lambda$ and the m*-r-fuzzy $g\circ F_\sigma$-interior of $\lambda$ are defined as follows:

1. $C_{m^*}(\lambda, r) = \{ \mu; \lambda \leq \mu, \mu$ is m*-r-fuzzy $g\circ$-closed $G_\delta \}$.
2. $I_{m^*}(\lambda, r) = \{ \mu; \lambda \geq \mu, \mu$ is m*-r-fuzzy $g\circ$-open $F_\sigma \}$.

Remark 2.1 Let $(X, T)$ be a smooth fuzzy topological space, $r \in I_0$. For any $\lambda \in I^X$, if $m_X^* = T$, then

1. $C_{m^*}(\lambda, r) = C_T(\lambda, r)$.
2. $I_{m^*}(\lambda, r) = I_T(\lambda, r)$.
Notation 2.2 Let \((X, T)\) be a smooth fuzzy topological space with an \(m^*\)-structure determined by \(T\). For \(r \in I_0\), any \(\lambda \in I^X\) which is both \(m^*\)-fuzzy \(\tilde{g}\)-open \(F_\sigma\) and \(m^*\)-fuzzy \(\tilde{g}\)-closed \(G_\delta\) is denoted by \(m^*\)-fuzzy \(\tilde{g}\)-COGF.

Definition 2.5 Let \((X, T)\) be a smooth fuzzy topological space with an \(m^*\)-structure \(m_X^*\) determined by \(T\) and let \(m_X^*\) satisfy property B. The \(m^*\)-smooth fuzzy space \((X, m^*)\) is said to be \(m^*\)-fuzzy basically disconnected if the \(m^*\)-fuzzy \(\tilde{g}\)-open \(F_\sigma\) set is \(m^*\)-fuzzy \(\tilde{g}\)-open \(F_\sigma\), \(r \in I_0\).

Proposition 2.1 For a smooth fuzzy topological space with an \(m^*\)-structure on \(X\) determined by \(T\) where \(m_X^*\) satisfies property B, the following conditions are equivalent:

(a) \((X, m^*)\) is an \(m^*\)-fuzzy basically disconnected space, \(r \in I_0\).

(b) For each \(m^*\)-fuzzy \(\tilde{g}\)-open \(F_\sigma\) set \(\lambda\), \(I_{m^*}(\lambda, r)\) is \(m^*\)-fuzzy \(\tilde{g}\)-closed \(G_\delta\), \(r \in I_0\).

(c) For each \(m^*\)-fuzzy \(\tilde{g}\)-open \(F_\sigma\) set \(\lambda\),
\[
C_{m^*}(\lambda, r) + C_{m^*}(\{1 - C_{m^*}(\lambda, r)\}, r) = \tilde{1}, \quad r \in I_0.
\]

(d) For every pair of \(m^*\)-fuzzy \(\tilde{g}\)-open \(F_\sigma\) sets \(\lambda\) and \(\mu\) with \(C_{m^*}(\lambda, r) + \mu = 1\), we have \(C_{m^*}(\lambda, r) + C_{m^*}(\mu, r) = \tilde{1}\), \(r \in I_0\).

Proof: (a) \(\Rightarrow\) (b). Let \(\lambda \in I^X\) be any \(m^*\)-fuzzy \(\tilde{g}\)-closed \(G_\delta\) set. Then, \(\tilde{1} - \lambda\) is \(m^*\)-fuzzy \(\tilde{g}\)-open \(F_\sigma\). Now, \(C_{m^*}(\{1 - C_{m^*}(\lambda, r)\}, r) = \{1 - C_{m^*}(\lambda, r)\}\), \(r \in I_0\). By (a), \(C_{m^*}(\{1 - C_{m^*}(\lambda, r)\}, r)\) is \(m^*\)-fuzzy \(\tilde{g}\)-open \(F_\sigma\), which implies that \(I_{m^*}(\lambda, r)\) is \(m^*\)-fuzzy \(\tilde{g}\)-closed \(G_\delta\).

(b) \(\Rightarrow\) (c). Let \(\lambda\) be any \(m^*\)-fuzzy \(\tilde{g}\)-open \(F_\sigma\) set. Then,
\[
C_{m^*}(\lambda, r) + C_{m^*}(\{1 - C_{m^*}(\lambda, r)\}, r)
\]
\[
= C_{m^*}(\lambda, r) + C_{m^*}(\{1 - C_{m^*}(\lambda, r)\}, r).
\]

(2.1)
Since \(\lambda\) is \(m^*\)-fuzzy \(\tilde{g}\)-open \(F_\sigma\), \(\tilde{1} - \lambda\) is \(m^*\)-fuzzy \(\tilde{g}\)-closed \(G_\delta\). Hence by (b), \(I_{m^*}(\{1 - C_{m^*}(\lambda, r)\}, r)\) is \(m^*\)-fuzzy \(\tilde{g}\)-closed \(G_\delta\). Therefore, by (2.1),
\[
C_{m^*}(\lambda, r) + C_{m^*}(\{1 - C_{m^*}(\lambda, r)\}, r) = C_{m^*}(\{1 - C_{m^*}(\lambda, r)\}, r).
\]
\[
= C_{m^*}(\lambda, r) + \tilde{1} - C_{m^*}(\lambda, r) = \tilde{1}.
\]

Therefore, \(C_{m^*}(\lambda, r) + C_{m^*}(\{1 - C_{m^*}(\lambda, r)\}, r) = \tilde{1}\).

(c) \(\Rightarrow\) (d). Let \(\lambda\) and \(\mu\) be \(m^*\)-fuzzy \(\tilde{g}\)-open \(F_\sigma\) sets with

95
\[ C_m^* (\lambda, r ) + \mu = \bar{1}. \]  
(2.2)

Then by (c), we have,
\[ 1 = C_m^* (\lambda, r ) + C_m^* (1 - C_m^* (\lambda, r ), r ) \]
\[ = C_m^* (\lambda, r ) + C_m^* (\mu, r ) \]
Therefore, \( C_m^* (\lambda, r ) + C_m^* (\mu, r ) = 1 \).

(d) \Rightarrow (a). Let \( \lambda \) be \( m^* \)-r-fuzzy \( g^* \)-open \( F_\sigma \) set. Put \( \mu = 1 - C_m^* (\lambda, r ) \). Then \( C_m^* (\lambda, r ) + \mu = 1 \). Therefore, by (d), \( C_m^* (\lambda, r ) + C_m^* (\mu, r ) = 1 \). This implies that \( C_m^* (\lambda, r ) \) is \( m^* \)-r-fuzzy \( g^* \)-open \( F_\sigma \) and so \((X, T)\) is \( m^* \)-fuzzy basically disconnected.

Proposition 2.2 Let \((X, T)\) be a smooth fuzzy topological space with an \( m^* \)-structure \( m_X^* \) determined by \( T \) and let \( m_X^* \) satisfy property B. Then \((X, m^* \) \) is \( m^* \)-fuzzy basically disconnected if and only if for all \( m^* \)-r-fuzzy \( g^* \)-open \( F_\sigma \) sets \( \lambda \) and \( m^* \)-r-fuzzy \( g^* \)-closed \( G_\delta \) set \( \mu \) such that \( \lambda \leq \mu \), \( C_m^* (\lambda, r ) \leq I_m^* (\mu, r ), r \in I_0 \).

Proof: Let \( \lambda \) be \( m^* \)-r-fuzzy \( g^* \)-open \( F_\sigma \) and \( \mu \) be \( m^* \)-r-fuzzy \( g^* \)-closed \( G_\delta \) with \( \lambda \leq \mu \). Then by (b) of Proposition 2.1, \( I_m^* (\mu, r ) \) is \( m^* \)-r-fuzzy \( g^* \)-closed \( G_\delta \). Also, since \( \lambda \) is \( m^* \)-r-fuzzy \( g^* \)-open \( F_\sigma \), \( C_m^* (\lambda, r ) \leq I_m^* (\mu, r ) \). Conversely, let \( \mu \) be any \( m^* \)-r-fuzzy \( g^* \)-closed \( G_\delta \) set. Then, \( I_m^* (\mu, r ) \in I_X^* \) is \( m^* \)-r-fuzzy \( g^* \)-open \( F_\sigma \) and \( I_m^* (\mu, r ) \leq \mu \). Therefore by assumption, \( C_m^* (I_m^* (\mu, r ), r ) \leq I_m^* (\mu, r ) \). This implies that \( I_m^* (\mu, r ) \) is \( m^* \)-r-fuzzy \( g^* \)-closed \( G_\delta \). Hence by (b) of Proposition 2.1, it follows that \((X, m^* \) \) is \( m^* \)-fuzzy basically disconnected.

Remark 2.2 Let \((X, m^* \) \) be any \( m^* \)-fuzzy basically disconnected space. Let \( \{ \lambda_i, \bar{1} - \mu_i / i \in N \} \) be a collection such that \( \lambda_i \)'s are \( m^* \)-r-fuzzy \( g^* \)-open \( F_\sigma \) and \( \mu_i \)'s \( m^* \)-r-fuzzy \( g^* \)-closed \( G_\delta \) and let \( \lambda \) and \( \mu \) be \( m^* \)-r-fuzzy \( g^* \)-COGF sets. If \( \lambda_i \leq \lambda \leq \mu_i \) and \( \lambda_i \leq \mu \leq \mu_i \) for all \( i, j \in N \), then there exists an \( m^* \)-r-fuzzy \( g^* \)-COGF set \( \gamma \) such that \( C_m^* (\lambda_i, r ) \leq \gamma \leq I_m^* (\mu_i, r ), \) for all \( i, j \in N, r \in I_0 \).

Proof: By Proposition 2.2, \( C_m^* (\lambda_i, r ) \leq C_m^* (\lambda, r ) \land I_m^* (\mu, r ) \leq I_m^* (\mu_i, r ), for all \( i, j \in N \). Therefore, \( \gamma = C_m^* (\lambda, r ) \land I_m^* (\mu, r ) \) is an \( m^* \)-r-fuzzy \( g^* \)-COGF set satisfying the required conditions.

Proposition 2.3 Let \((X, m^* \) \) be any \( m^* \)-fuzzy basically disconnected space. Let \( \{ \lambda_i \}_{i \in Q} \) and \( \{ \mu_i \}_{i \in Q} \) be monotone increasing collections of \( m^* \)-r-fuzzy \( g^* \)-open \( F_\sigma \) sets and \( m^* \)-r-fuzzy \( g^* \)-closed \( G_\delta \) sets of \((X, m^* \) \) and suppose that \( \lambda_{q_1} \leq \mu_{q_2} \) whenever \( q_1 < q_2 \) \( (Q \) is
the set of all rational numbers). Then there exists a monotone increasing collection \( \{ \gamma_l \} \in Q \) of \( m^* \) r-fuzzy \( \tilde{g} \)-COGF sets of \( (X, m^*) \) such that \( C_{m^*}(\lambda_{q1}, r) \leq \gamma_{q2} \) and \( \gamma_{q1} \leq I_{m^*}(\mu_{q2}, r) \) whenever \( q_1 < q_2, r \in I_0 \).

Proof: Consider the families \( \{ C_{m^*}(\lambda_{q}, r) \} \) and \( \{ I_{m^*}(\mu_{q}, r) \} \). By Proposition 2.2, \( C_{m^*}(\lambda_{q1}, r) \leq I_{m^*}(\mu_{q2}, r) \) if \( q_1 \leq q_2 \). Now, applying the argument used in Lemma 3.6 of [13] and the above remark, we get the proof.

3. Properties and characterizations of \( m^* \)-fuzzy basically disconnected spaces

In this section, the concepts of Lower \( m^* \)-fuzzy continuous functions and upper \( m^* \)-fuzzy continuous functions are introduced. In this regard, various properties and characterizations are discussed.

Definition 3.1 Let \( (X, T) \) be a smooth fuzzy topological space with an \( m^* \)-structure \( m_X^* \) determined by \( T \) and let \( m_X^* \) satisfy property B. A function \( f : X \rightarrow R (I) \) is called lower (resp. upper) \( m^* \)-fuzzy continuous if \( f^{-1}(R_t) \) (resp. \( f^{-1}(L_t) \)) is \( m^* \) r-fuzzy \( \tilde{g} \)-open \( F_\sigma \) (resp. \( m^* \) r-fuzzy \( \tilde{g} \)-open \( F_\sigma / m^* \) r-fuzzy \( \tilde{g} \)-closed \( G_\delta \)), for each \( t \in R, r \in I_0 \).

Proposition 3.1 Let \( (X, T) \) be a smooth fuzzy topological space with an \( m^* \)-structure \( m_X^* \) determined by \( T \) and let \( m_X^* \) satisfies property B. For \( \lambda \in I_X \), and \( r \in I_0 \), let \( f : X \rightarrow R (I) \) be such that

\[
f(x)(t) = \begin{cases} 
1 & \text{if } t < 0 \\
\lambda(x) & \text{if } 0 \leq t \leq 1 \\
0 & \text{if } t > 0 
\end{cases}
\]

for all \( x \in X \). Then \( f \) is lower (resp. upper) \( m^* \)-fuzzy continuous iff \( \lambda \) is \( m^* \) r-fuzzy \( \tilde{g} \)-open \( F_\sigma \) (resp. \( m^* \) r-fuzzy \( \tilde{g} \)-open \( F_\sigma / m^* \) r-fuzzy \( \tilde{g} \)-closed \( G_\delta \)), \( r \in I_0 \).

Definition 3.2 Let \( (X, T) \) be a smooth fuzzy topological space with an \( m^* \)-structure \( m_X^* \) determined by \( T \) and let \( m_X^* \) satisfy property B. The characteristic function of \( \lambda \in I_X \) is the function \( 1_\lambda : X \rightarrow I_X \) defined by

\[
1_\lambda(x) = \lambda(x), \quad x \in X, \quad r \in I_0.
\]

Proposition 3.2 Let \( (X, T) \) be a smooth fuzzy topological space with an \( m^* \)-structure \( m_X^* \) determined by \( T \) and let \( m_X^* \) satisfy property B; let \( \lambda \in I_X \). Then \( 1_\lambda \) is lower (resp. upper) \( m^* \)-fuzzy continuous iff \( \lambda \) is \( m^* \) r-fuzzy \( \tilde{g} \)-open \( F_\sigma \) (resp. \( m^* \) r-fuzzy \( \tilde{g} \)-open \( F_\sigma / m^* \) r-fuzzy \( \tilde{g} \)-closed \( G_\delta \)), \( r \in I_0 \).

Proof: The proof follows from the Proposition 3.1.
Definition 3.3 Let \((X, T)\) and \((Y, S)\) be any two smooth fuzzy topological spaces with the respective m*-structures \(m_1^*\) and \(m_2^*\) determined by \(T\) and \(S\) respectively and let both \(m_1^*\) and \(m_2^*\) satisfy property B. A function \(f: (X, m_1^*) \to (Y, m_2^*)\) is called strongly m*-fuzzy continuous if \(f^{-1}(\lambda) \in I^X\) is m*-r-fuzzy \(g\)-open \(F_\sigma\) for every m*-r-fuzzy \(g\)-closed \(G_\delta\), for every m*-r-fuzzy \(g\)-open \(F_\sigma\) set \(\lambda \in I^Y\), \(r \in I_0\).

Proposition 3.3 Let \((X, T)\) be a smooth fuzzy topological space with an m*-structure \(m_X^*\) determined by \(T\) and let \(m_X^*\) satisfy property B. Then for \(r \in I_0\), the following conditions are equivalent:

(a) \((X, m^*)\) is an m*-fuzzy basically disconnected space.

(b) If \(g, h: X \to R(I)\) where \(g\) is lower m*-fuzzy continuous, \(h\) is upper m*-fuzzy continuous, then there exists \(f \in C_{Sm^*}(X, m^*)\) such that \(g \leq f \leq h\). \([C_{Sm^*}(X, m^*) = \text{collection of all strongly m*-fuzzy continuous functions on } X \text{ with values in } R(I)]\).

(c) If \(\tilde{1} - \lambda, \mu\) are m* r-fuzzy \(g\)-open \(F_\sigma\) sets such that \(\mu \leq \lambda\), then there exists a strongly m*-fuzzy continuous \(f: X \to I^X\) such that \(\mu \leq (\tilde{1} - L_1)f \leq R_0f \leq \lambda\).

Proof : (a) \(\Rightarrow\) (b). Define \(H_k = L_k h\) and \(G_k = (\tilde{1} - R_k)g\), \(k \in Q\). Thus we have two monotone increasing families of m* r-fuzzy \(g\)-open \(F_\sigma\) sets and m* r-fuzzy \(g\)-closed \(G_\delta\) sets of \((X, m^*)\) respectively. Moreover \(H_k \leq G_s\) if \(k < s\). By Proposition 2.3, there exists a monotone increasing family \(\{F_k\}_{k \in Q}\) of m* r-fuzzy \(g\)-COGF sets of \((X, m^*)\) such that \(C_{m^*}(H_k, r) \leq F_s\) and \(F_k \leq I_{m^*}(G_s, r)\) whenever \(k < s\). As in the proof of Proposition 3.7 of [13], let us define \(V_t = \bigwedge_{k < t}(\tilde{1} - F_k)\) for all \(t \in R\) and again as in [13], one can show that \(f: X \to R(I), f(x)(t) = V_t(x)\) is well defined and has the required properties.

(b) \(\Rightarrow\) (c). Suppose that \(\lambda\) is m* r-fuzzy \(g\)-closed \(G_\delta\) and \(\mu\) is m* r-fuzzy \(g\)-open \(F_\sigma\) such that \(\mu \leq \lambda\). Then, \(1_\mu \leq 1_\lambda\) where \(1_\mu, 1_\lambda\) are lower and upper m*-fuzzy continuous functions respectively. Hence by (b), there exists a strong m*-fuzzy continuous function \(f: X \to R(I)\) such that, \(1_\mu \leq f \leq 1_\lambda\). Clearly, \(f(x) \in I^X\), for all \(x \in X\) and \(\mu = (\tilde{1} - L_1)1_\mu \leq (\tilde{1} - L_1)f \leq R_0f \leq R_01_\lambda = \lambda\). Therefore, \(\mu \leq (\tilde{1} - L_1)f \leq R_0f \leq \lambda\).
(c) ⇒ (a). \((\overline{I} - L_{\ell})\) and \(R_0 f\) are \(m^*\) r-fuzzy \(\tilde{g}\)-COGF sets. By Proposition 2.2, \((X, m^*)\) is an \(m^*\)-fuzzy basically disconnected space.

4. Tietze Extension Theorem

In this section, Tietze Extension Theorem for \(m^*\)-fuzzy basically disconnected space is studied.

Proposition 4.1 Let \((X, m^*)\) be an \(m^*\)-fuzzy basically disconnected space and let \(A \subset X\) be such that \(1_A\) is \(m^*\) r-fuzzy \(\tilde{g}\)-open \(F_\sigma\). Let \(f : (A, m^*/A) \to \tilde{I}^X\) be strong \(m^*\)-fuzzy continuous. Then \(f\) has a strong \(m^*\)-fuzzy continuous extension over \((X, m^*), r \in I_0\).

Proof: Let \(g, h : X \to \tilde{I}^X\) be such that \(g = f = h\) on \(A\) and \(g(x) = 0, h(x) = 1\) if \(x \notin A\).

We now have

\[
R_t g = \begin{cases} \mu_t \wedge 1_A & \text{if } t \geq 0 \\ 1 & \text{if } t < 0 \end{cases}
\]

where \(\mu_t\) is \(m^*\) r-fuzzy \(\tilde{g}\)-open \(F_\sigma\) and is such that \(\mu_t/A = R_t f\) and

\[
L_t h = \begin{cases} \lambda_t \wedge 1_A & \text{if } t \leq 1 \\ 1 & \text{if } t > 1 \end{cases}
\]

where \(\lambda_t\) is \(m^*\) r-fuzzy \(\tilde{g}\)-COGF and is such that \(\lambda_t/A = L_t f\). Thus, \(g\) is lower \(m^*\)-fuzzy continuous and \(h\) is upper \(m^*\)-fuzzy continuous with \(g \leq h\). By Proposition 3.3, there is a strong \(m^*\)-fuzzy continuous function \(F : X \to \tilde{I}^X\) such that \(g \leq F \leq h\).

Hence \(F \equiv f\) on \(A\).

REFERENCES


