Petrov Invariants for 1-D Control Hamiltonian Systems

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Abstract

In this paper we consider the action of symplectic feedback transformations on 1-D control Hamiltonian systems. We study differential invariants of the pseudogroup of feedback symplectic transformations, which we call Petrov invariants, and show that the algebra of invariants possesses a natural Poisson structure and central derivations. This structure allows us to classify regular 1-D control Hamiltonian systems.

Key Words: control Hamiltonian systems, differential invariants, Lie pseudogroups, symplectic feedback transformation, Poisson structures, Petrov invariant.

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1 Feedback Transformations and Control Hamiltonian Systems

A 1-D control Hamiltonian system with a Hamiltonian $H = H(q, p, u)$ is given by vector field

$$H_p \partial_q - H_q \partial_p,$$

where $q$ and $p$ are the phase variables, and $u$ is a control parameter.

In control theory it is common to call transformations of the form

$$(q, p, u) \mapsto (Q(q, p), P(q, p), U(q, p, u)),$$

as feedback transformations (see [1, 3, 5, 8, 9]).

In our case they should preserve the class of Hamiltonian systems. Hence, it is easy to check, that they are of the following special form:

$$(q, p, u) \mapsto (Q(q, p), P(q, p), U(u)),$$  \hspace{1cm} (1.2)

where $(q, p) \mapsto (Q(q, p), P(q, p))$ are symplectic transformations.

Such transformations we call symplectic feedback transformations.

We’ll consider the problem of symplectic feedback equivalence of systems (1.1) with respect to transformations (1.2).

Remark that these transformations act on the Hamiltonians in the natural way:

$$\varphi^* : H(Q, P, U) \mapsto H(Q(q, p), P(q, p), U(u)).$$

2 Control Systems’ Bundle

Let $M = \mathbb{R}^2$ be a phase space and let

$$\Omega = dp \wedge dq$$

be the structure 2-form on $M$.

Consider an extended phase space $B = M \times \mathbb{R}$ with coordinates $q, p, u$.

Infinitesimal symplectic feedback transformations are vector fields on the space $B$ of the form

$$X_{H, \lambda} = X_H + Y_\lambda$$  \hspace{1cm} (2.1)

where

$$X_H = \frac{\partial H}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial}{\partial p},$$

and

$$Y_\lambda = \lambda(u) \frac{\partial}{\partial u},$$
and $H = H(p, q)$.

The Lie pseudogroup of symplectic feedback transformations we denote by $G$ and the corresponding Lie algebra of symplectic feedback vector fields will be denoted by $G$.

Let

$$
\pi : B \times \mathbb{R} \to B, \quad \pi : (q, p, u, h) \mapsto (q, p, u).
$$

be one-dimensional trivial bundle over $B$.

Sections of this bundle can be viewed as a functions of the form $f(q, p, u)$, i.e. functions that define the control Hamiltonian systems.

For this reason, we call $\pi$ as control system bundle.

Let $J^k(\pi)$ be the space of $k$-jets of sections of the bundle $\pi$.

Denote by $q, p, u, h, h_\sigma$ the canonical coordinates on $J^k(\pi)$.

Here $\sigma$ are multi-indexes of length $\leq k$:

$$
\sigma = (\sigma_1, \sigma_2, \sigma_3), \quad |\sigma| = \sigma_1 + \sigma_2 + \sigma_3 \leq k.
$$

Let $h = H(q, p, u)$ be a section of the bundle $\pi$. Then, in canonical coordinates, $k$-jet at a point $a \in B$ of this section has the form

$$
[H]^k_a = \left( x(a), H(a), \ldots, \frac{\partial^{|\sigma|} H}{\partial x^\sigma}(a), \ldots \right),
$$

where $|\sigma| \leq k$ and $x = (q, p, u)$.

Prolongations of a vector field $X$ and a transformation $\varphi$ into the spaces $J^k(\pi)$ will be denoted by $X^{(k)}$ and $\varphi^{(k)}$ respectively.

### 3 Petrov Differential Invariants

A smooth function $J$ on $k$-jet space $J^k(\pi)$, which rational in fibrewise variables $h_\sigma$, we call Petrov invariant of order $\leq k$, if

$$
(\varphi^{(k)})^* (J) = J \quad (3.1)
$$

for any symplectic feedback transformation $\varphi$, or

$$
X_{H,\lambda}^{(k)}(J) = 0 \quad (3.2)
$$

for any symplectic feedback vector field $X_{H,\lambda}$.

Remark that vector fields $X_{H,\lambda}^{(k)}$ generate a completely integrable distribution on $J^k(\pi)$ and rational first integrals of this distribution are Petrov differential invariants.

In a similar way, a function $J$ on $J^k(\pi)$ is called a relative Petrov invariant of order $\leq k$, if

$$
X^{(k)}(J) = \lambda_X J, \quad (3.3)
$$
for any symplectic feedback vector field $X$ and a weight 1-cocycle

$$\lambda : X \in \mathcal{G} \mapsto \lambda_X \in C^\infty (J^k\pi),$$

(3.4)
on the Lie algebra $\mathcal{G}$.

A total derivation

$$\nabla = A \frac{d}{dq} + B \frac{d}{dp} + C \frac{d}{du},$$

(3.5)
is called an invariant derivation if it commutes with any symplectic feedback vector field, i.e. if the following diagram

$$\begin{array}{ccc}
C^\infty (J^\infty (\pi)) & \xrightarrow{\nabla} & C^\infty (J^\infty (\pi)) \\
X^{(\infty)} \downarrow & & \downarrow X^{(\infty)} \\
C^\infty (J^\infty (\pi)) & \xrightarrow{\nabla} & C^\infty (J^\infty (\pi))
\end{array}$$

commutes, for any vector field $X \in \mathcal{G}$.

Here $A, B,$ and $C$ are fibrewise rational smooth function on the space $J^\infty (\pi)$ and $\frac{d}{dx}$ are operators of the total derivatives in $x$ (see [6]).

### 4 Dimensions of Jet Orbits

Splitting $B = M \times \mathbb{R}$ gives the decomposition

$$J^k_b (\pi) = \bigoplus_{s=0}^k J^{k-s}_a (M)$$
of the jet space at a point $b = (a, 0) \in B, a \in M$, in the following way.

Each function $f (q, p, u)$ can be presented in the following form

$$f = f_0 (q, p) + u f_1 (q, p) + \cdots + \frac{f_s (q, p)}{s!} u^s + \cdots + \frac{f_k (q, p)}{k!} u^k + u^{k+1} g (q, p, u),$$

where $f_0, \ldots, f_k, g$ are smooth function.

Therefore, for $k$-jets we get the following decomposition

$$[f]^{k}_b = [f_0]^{k}_a \oplus [f_1]^{k-1}_a \oplus \cdots \oplus [f_k]^{0}_a.$$

To find codimensions of $G$-orbits in $J^k (\pi)$ we remark that $G$ acts in transitive way on $B$.

Therefore, these codimensions are equal to codimensions of the $G_b$-orbits in the fibre $J^k_b (\pi)$, where $G_b$ is the stabilizer of the point $b$ in $G$.

Let $\mathcal{O} (x_k) = G^{(k)}_b (x_k)$ be the orbit of $x_k = [f]^{k}_b$. 

92
Then the tangent space to the orbit at the point \(x_k\) is generated by values of vector fields \(X^{(k)}_{H,\lambda}\) at the point, where \(H\) has 2-nd order at the point \(a\), and \(\lambda(0) = 0\).

In other words,

\[
H \in \mu_a^2, \lambda \in \mu_0,
\]

where \(\mu_a\) and \(\mu_0\) are the maximal ideals of the points \(a \in M\) and \(0 \in \mathbb{R}\).

The general prolongation formula (see, for example, [6]) shows that, in this case, value of \(X^{(k)}_{H,\lambda}\) at the point \(x_k\) equals to

\[
[X_H(f) + \lambda(u) \ f_u]_b^k.
\]

Using the above decomposition we write \(s\)-component of this vector in the form

\[
[X_H(f_s)]_a^{k-s} + \sum_{i=1}^{k} \left(\begin{array}{c}s \\ i \end{array}\right) \lambda_i [f_{s-i+1}]_a^{k-s},
\]

where \(\lambda_i = \lambda^{(i)}(0)\).

Consider the correspondence

\[(H, \lambda) \mapsto [X_H(f) + \lambda(u) \ f_u]_b^k\]

as a linear operator

\[
\kappa_k : J^{k+1,1}_a(M) \oplus J^{k,0}_0(\mathbb{R}) \to J^k_b(\pi).
\]

Here we denoted by \(J^{k+1,1}_a(M)\) the kernel of the projection \(J^{k+1}_a(M) \to J^1_a(M)\), and by \(J^{k,0}_0(\mathbb{R})\) the kernel of the projection \(J^k_b(\mathbb{R}) \to J^0_b(\mathbb{R})\).

We say that the point \(x_k \in J^k_b(\pi)\) is regular, if \(f_1(a) \neq 0\) and vectors \(X_{f_0}(a)\) and \(X_{f_1}(a)\) are linear independent.

**Theorem 4.1.** Let \(x_k \in J^k_b(\pi)\) be a regular point. Then

- \(\dim \ker(\kappa_k) = 1\).
- **Codimension of the orbit** \(G^{(k)}_b(x_k)\) **is equal to**

\[
\frac{k(k + 5)(k - 2)}{6} + 2.
\]

**Proof.** The kernel consist of solutions of the following linear system

\[
E_s = [-X_{f_s}(H)]_a^{k-s} + \sum_{i=0}^{s} \left(\begin{array}{c}s \\ i \end{array}\right) \lambda_i [f_{s-i+1}]_a^{k-s} = 0
\]

where \(s = 0, ..., k\).
Taking 0-jets of $E_s$, and taking in account that $H \in \mu_a^2$, we get the following system

$$\sum_{i=0}^{s} \binom{s}{i} \lambda_i f_{s-i+1}(a) = 0,$$

which has the only trivial solution, if

$$f_1(a) \neq 0.$$

Assuming that the last condition holds we get the following linear system for $k$-jet $H$:

$$E^0_s = [X_{f_s}(H)]^{k-s} = 0,$$

where $s = 0, \ldots, k-1$.

Taking now 1-jets of $E^0_s$ we get the following system

$$[X_{f_s}(H)]^1_a = 0,$$

where $s = 0, \ldots, k-1$.

Let $\theta_2 = [H]^2_a \in S^2 T^*_a$, and denote by $\delta : S^l(T^*_a) \to S^{l-1}(T^*_a) \otimes T^*_a$ the Spencer $\delta$-operator.

Then the last equations can be rewritten as follows

$$X_{f_s,a}] \delta(\theta_2) = 0.$$

Therefore, if $k \geq 2$ and vectors $X_{f_s,a}$ are linear independent, we get $\delta(\theta_2) = 0$, and $\theta_2 = 0$, or $H \in \mu_a^3$.

Then the projections of $E^0_s$ into 2-nd jets give us the next linear system

$$[X_{f_s}(H)]^2_a = 0,$$

for $s = 0, \ldots, k-2$, or

$$X_{f_s,a}] \delta(\theta_3) = 0,$$

where $\theta_3 = [H]^3_a \in S^3 T^*_a$.

Assuming once more that $k \geq 3$, and that vectors $X_{f_s,a}$ are linear independent, we get $\delta(\theta_3) = 0$, and $\theta_3 = 0$, or $H \in \mu_a^4$.

Continue in the same way we arrive to the condition $H \in \mu_a^{k+1}$ and to linear system

$$X_{f_0,a}] \delta(\theta_{k+1}) = 0,$$

$$\theta_{k+1} = [H]^{k+1}_a \in S^{k+1} T^*_a.$$

The last system has 1-dimensional solution space. \hfill \Box

**Corollary 1.** Rational Petrov invariants of order $\leq k$ form a field. The transcendence degree of this field equals to

$$\nu_k = \frac{k(k+5)(k-2)}{6} + 2.$$
Corollary 2. There are \( \nu_k \) independent Petrov invariants of order \( \leq k \).

The first values of \( \nu_k \) given in the following table:

\[
\begin{array}{cccccc}
  k & 1 & 2 & 3 & 4 & 5 \\
  \nu_k & 1 & 2 & 6 & 12 & 25 \\
\end{array}
\]

5 Petrov Invariants of low order

In this section we describe Petrov invariants in order \( \leq 3 \). In order \( \leq 2 \) the result is rather obvious but in order 3 it was found by Ian Anderson’s Differential Geometry package in Maple.

Indeed, we have obvious Petrov invariant of order 0,

\( J_0 = h. \)

Moreover, in order 1 function \( h_u \) and the total derivation

\[
\frac{d}{du}
\]

are relative invariants.

In order 2 the function

\[
(h, h_u) = h_p h_{uq} - h_q h_{up}
\]

is a relative invariant too.

Compare their weights we find the following Petrov invariants

\[
\begin{align*}
J_0 &= h, \\
J_2 &= \frac{h_p h_{uq} - h_q h_{up}}{h_u}
\end{align*}
\]

and invariant derivation

\[
\nabla = \frac{1}{h_u} \frac{d}{du}.
\]

To find invariants of order three we remark that the above corollary shows that in addition to invariants \( J_0, J_2 \) we have four invariants of pure order three.
Solving in Maple equation (3.2) for \( k = 3 \), we get:

\[
J_{30} = \frac{1}{h^3_u}(h_qh_uh_{pua} - h_ph_uh_{qua} - h_qh_{puu}h_{ua} + h_pqhu_{ua}),
\]

\[
J_{31} = \frac{1}{h^2_u}(h_q^2h_{ppu} - 2h_qh_{ppu} + h^2_qh_{pp} - h_qh_{qu}h_{pp} + h_qh_{pq}h_{pu} - h_ph_{pq}h_{pp} + h_pqhu_{pp} - h^2_puh_{pp}),
\]

\[
J_{32} = \frac{1}{h^2_u}(h_qh_{qu}h_{ppu} - (h_ph_{pu} + h_ph_{qu})h_{ppu} = h_pqhu_{pp} - h^2_puh_{qu} - 2h_ph_{qu}h_{pp} - h^2_qh_{pp}),
\]

\[
J_{33} = \frac{1}{h^3_u}(h_{pu}h_{quu} - h_{qu}h_{puu}).
\]

Note also that the invariant \( J_{30} \) we can get from the invariant \( J_2 \) by differentiation: \( J_{30} = \nabla(J_2) \).

These computations show that invariants up to order 3 are polynomials in \( h_\sigma, h_u^{-1} \). For this reason, from now on we call Petrov invariants such differential invariants of the symplectic feedback pseudogroup, which are polynomials in \( h_\sigma, h_u^{-1} \).

To find Petrov invariants of higher order we’ll need an additional structure on the algebra of invariants.

### 6 Poisson Algebra Structure

Let us consider the structure form \( \Omega \) as a horizontal form on \( J^\infty(\pi) \), and let’s try to repeat the construction of the Hamiltonian vector fields.

Take a function \( A \in C^\infty(J^\infty(\pi)) \) and let’s try to find a total derivation \( X_A \) such that \( X_A|\Omega = \hat{d}A \).

Because \( \nabla|\Omega = 0 \) one should correct the righthand side in such a way that it will annihilate derivation \( \nabla \).

Such correction leads us to the following result.

**Theorem 6.1.** 1. Let \( A \) be a smooth function on \( J^\infty(\pi) \), \( A \in C^\infty(J^\infty(\pi)) \). Then relations

\[
X_A|\Omega = \hat{d}A - \nabla(A)\hat{d}h,
\]

\[
X_A(A) = 0,
\]

define a unique total derivation \( X_A \) on \( J^\infty(\pi) \).

2. In canonical coordinates \( X_A \) on \( J^\infty(\pi) \) has the following form:

\[
X_A = \left( \frac{dA}{dp} - \nabla(A)h_p \right) \frac{d}{dq} - \left( \frac{dA}{dq} - \nabla(A)h_q \right) \frac{d}{dp} + \left( \frac{dA}{dq}h_p - \frac{dA}{dp}h_q \right) \nabla.
\]
3. If $A$ is a feedback differential invariant, then $X_A$ is an invariant derivation.

Therefore, if $A$ and $B$ are Petrov invariants, then the function $X_A(B)$ is so also.

Let’s introduce the following bracket on the algebra of Petrov invariants:

$$[A, B] = X_A(B).$$

(6.1)

This bracket can be rewritten as

$$[A, B] = (A, B) - \nabla (A)(h, B) + \nabla (B)(h, A),$$

where

$$(A, B) = \frac{dA}{dp} \frac{dB}{dq} - \frac{dA}{dq} \frac{dB}{dp}$$

is the prolongation of the classical Poisson bracket to $J^\infty(\pi)$.

**Theorem 6.2.** 1. Algebra of Petrov invariants is Poisson with respect to bracket (6.1).

2. The operator $\nabla$ is a derivation in this algebra:

$$\nabla[A, B] = [\nabla A, B] + [A, \nabla B].$$

3. The differential invariant $J_0$ is a Casimir function in the Poisson algebra, i.e. $[A, J_0] = 0$ for any Petrov invariant $A$.

7 Structure of the Petrov Invariant Algebra

Recall that a point $x_k = [f]^k_b \in J^k_b (\pi)$ is regular if $f_u(b) \neq 0$, and vectors $X_{f_u}$ and $X_{f_u,b}$ are linear independent.

Orbits $O(x_k)$ of regular points we call regular.

The above discussion together with the final classification theorem (see below) shows that the following result holds.

**Theorem 7.1.** Algebra of Petrov invariants, as a Poisson algebra, is generated by the invariants $J_0, J_2, J_{30}, J_{31}, J_{32}, J_{33}$, and invariant derivation $\nabla$. This algebra separates regular orbits.

8 Feedback classification

Consider a control Hamiltonian system given by a Hamiltonian $H(q, p, u)$, and denote by $A_H$ the value of a Petrov invariant $A$ on $H$. 
We say that the control system is *regular* in a domain $D \subset B$, if there are two Petrov invariants, say $A$ and $B$, such that functions

$$H = h_H, A_H, B_H$$

are independent in the domain, and the bracket

$$[A, B]_H \neq 0$$

in the domain.

Such invariants $A$ and $B$ we’ll call *basic* for the system.

**Lemma 8.1.** Let

$$\hat{d}A \wedge \hat{d}B \wedge \hat{d}h \neq 0, \quad [A, B] \neq 0$$

in a domain of $J^\infty (\pi)$.

Then in this domain we have the following representation of the structure form:

$$\Omega = \left( \nabla (B) \hat{d}A - \nabla (A) \hat{d}B \right) \wedge \hat{d}h - \frac{1}{[A, B]} \hat{d}A \wedge \hat{d}B.$$  

Proof. Let

$$\Omega = \left( \alpha \hat{d}A + \beta \hat{d}B \right) + \gamma \hat{d}A \wedge \hat{d}B$$

in the domain.

Then

$$\nabla \Omega = \left( \alpha \nabla (A) + \beta \nabla (B) \right) \hat{d}h - \left( \alpha + \gamma \nabla (B) \right) \hat{d}A + \left( -\beta + \gamma \nabla (A) \right) \hat{d}B = 0.$$  

Therefore,

$$\alpha = \gamma \nabla (B), \quad \beta = -\gamma \nabla (A).$$

On the other hand, we have

$$X_A \Omega = \beta X_A (B) \hat{d}h - \gamma X_A (B) \hat{d}A = \hat{d}A - \nabla (A) \hat{d}h.$$  

Therefore,

$$\alpha = \frac{\nabla (B)}{[A, B]}, \quad \beta = -\frac{\nabla (A)}{[A, B]}, \quad \gamma = -\frac{1}{[A, B]}.$$  

\[ \square \]
Let now \( H \) be the Hamiltonian of a control system which is regular in a domain \( D \).

Then functions
\[
x \overset{\text{def}}{=} h_H, \\
y \overset{\text{def}}{=} A_H, \\
z \overset{\text{def}}{=} B_H,
\]
for the basic Petrov invariants \( A \) and \( B \) can be viewed as coordinates in \( D \).

Denote by \( P_0, P_1, P_2 \) the values of invariants \(-\frac{1}{[A,B]}, \frac{\nabla(B)}{[A,B]} \) and \(-\frac{\nabla(A)}{[A,B]} \) on \( H \) and call them defining functions for the system.

They are functions in \((x,y,z)\) and \( P_0 \neq 0 \).

The above lemma shows that in coordinates \((x,y,z)\) the structure form \( \Omega \) and vector field \( \nabla H \) has the following form:
\[
\Omega = (P_1 dy + P_2 dz) \wedge dx + P_0 dy \wedge dz, \\
\nabla H = \partial_x + \frac{P_2}{P_0} \partial_y - \frac{P_1}{P_0} \partial_z.
\]

This gives us immediately the following classification of regular control systems.

**Theorem 8.1.** Two regular control Hamiltonian systems are feedback equivalent if and only if they have the same basic invariants and the same defining functions.

**References**


